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A CLASS OF ADAPTIVE CONTROL SYSTEMS WITH SINUSOIDAL PARAMETER PERTURBATION

by

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CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

CHAPTER I
INTRODUCTION

In the design of an information or data-processing unit, there are two extremes which may be followed. The first is to design enormous versatility (and therefore great complexity) into the data processing unit. The other extreme is to design a data-processing unit to handle only a restricted class of input signals (thereby reducing complexity and increasing data handling speed). In such a case the sensor must be adjusted so that only this class of signals is present. A typical class of signals might consist of voltages between zero and one volt. For example, if the sensor is an optical one, then for overall accuracy, the sensor output must be relatively independent of ambient light conditions. This of course implies an automatic control of amplifier gain or lens opening so that the signal will fall between prescribed bounds. In the case of a radar system when one changes receiver gain, one should also modify pulse width and I. F. amplifier pass band in order to make optimum usage of the radar system. Otherwise noise may dominate in areas of small return or resolution may be impaired in regions of large return.

In general, an adaptive control system is needed at the sensor to maintain the sensor output within allowable limits if the data-processing unit is to be effective. Before such adaptive systems can

be adequately designed for use in the applications mentioned above, it is necessary that analysis methods, and ultimately, synthesis methods, be developed for these systems. The remainder of this report contains the development of analysis methods for a class of adaptive control systems. This class of systems is versatile enough that applications similar to the one above may be realized. A relatively simple system is used as an example throughout this report.

CHAPTER II
ADAPTIVE CONTROL

What is Adaptive Control?

The term adaptive control has been defined in many ways by many people. Webster defines adaptive as showing properties of making suitable; fitting; adjusting; control as to exercise directing, guiding, or restraining power over, or any device which performs the above.[5] Another version of the definition is a system which can determine its performance equations and the effect of its environment in sufficient detail that some performance criterion shall always be optimum in spite of arbitrary variations in the effect of the environment. Finally an adaptive control system may be defined as a control system which is inherently capable of maintaining a desired performance in the presence of a changing environment.[7] There are, of course, many other definitions, each of which is correct in the sense that there is at this writing no accepted standard. Furthermore, there exist presently many names for these systems: adaptive, self-optimizing, self-adjusting, and supervisory control systems, to name a few. Of these, the most descriptive seems to be supervisory, since that term best describes the operation of adjusting parameter values according to a performance criterion. The most popular, however, seems to be the term adaptive. The popularity of the word adaptive is enhanced to some extent by the relatively new field of bionics and

by the desire of some to identify the operation of this type of system with the performance of certain tasks by humans in which the human adapts himself to a varying environment. It is felt that this identification is a good one; it has been the policy of the author and his associates to study the human in order to learn from his remarkable adaptive capabilities how to construct better adaptive control systems.

The definition of the term adaptive control appears to be best stated as a combination of several definitions. That is, an adaptive control system is any system that adjusts its operating characteristics or internal parameters in such a manner that some performance criterion is optimized in the presence of a changing environment. Here the term environment is broadened to include physical environment, e.g., the air density in the vicinity of an airplane; disturbances, e.g., air turbulence in the vicinity of an airplane; and a statistical change at the system input, e.g., a shift in the mean frequency of Gaussian noise at the input.

How Does Adaptive Control Work?

There are several modes of system operation that fall into the broad classification of adaptive control. As stated before, however, all types of operation include a performance criterion which is to be optimized. One approach is to consider the system from a transient response point of view, applying a test signal to the input of the

system. Here the adjusted system is given a step or impulse input periodically, and its output is measured in terms of error squared, overshoot, settling time, or some other transient response measure, including comparison with the response of a model system. Then, based upon this measure, the system is adjusted to optimize the criterion (i.e., minimize settling time).

Another method is to perturb or dither a parameter and sense the effect of this perturbation. Again, this effect can be measured by comparing the output of the adjusted system to that of a model system and adjusting the parameter to reduce the difference (i.e., minimize the square of the error between the two outputs). This perturbation may be sinusoidal, random, step-wise discontinuous, etc.

Examples

Under the definition previously stated, several examples of adaptive control systems may be mentioned. In 1955, an optimizing servomechanism was built at the Ohio State University which could locate extrema on a slotted line.[9] In 1957, the University of California reported on in-flight evaluation of aerodynamic parameters by means of programmed disturbances.[10] In 1960, a system similar to the one analyzed herein was built at the Ohio State University to measure parameters of the human in a tracking task.[11] Here, the parameter was varied sinusoidally. Certain industries, the

petroleum industries specifically, are using adaptive control systems in their refineries to optimize system operation in order to maximize dollars profit! Thus it is not necessary that a system variable in the ordinary sense be used as a performance criterion.

When is Adaptive Control Advantageous?

This question may appear trivial, since it would seem that in most applications involving control systems, the most sophisticated system (i.e., the adaptive system) should give the best performance. This, however, is not the case. If, for example, the input to a system is a known function of time, the best response is found when using a cam arrangement between input and output. Here there is zero error. If the input is random but stationary, then the best system is a linear one[12] providing the criterion for the system response falls into the usual classification. That is, the output must be a linear function of the input, and the performance criterion must be minimum rms error. If, however, the environment (in the broad sense) changes with time, then, and only then, can an adaptive control system be used to advantage.

Nature of the Adaptive Control System in this Study

The system considered in this paper is the one previously described in terms of parameter dither where, in this case, sinusoidal dither is used. In particular, two systems in this class

are considered. The first system considered is the "slow" system. The term slow is used for the following reasons: the dither frequency is much smaller than the mean frequencies of the input signal and the responses of the adjusted and model systems are fast compared with the adaptive loop, so that the primary system can be considered to be in the steady state at all times during the adaption process. In the second case, the "fast" system is so named because the dither frequency is large with respect to the mean frequencies of the input, and the response of the adjusted system is dependent upon past parameter variations. The system operation is designed as follows: The difference in outputs of the model and adjusted system is squared and forced to zero. This is the performance criterion for both the slow and fast systems. To effect this forcing to zero, the average value of the adjusted parameter is driven in a direction so that the error squared is decreased. The operation of the two types is analyzed for the case of an input which is random in nature. The input is obtained by filtering a square wave with Poisson-distributed zero crossings.[13] The mean frequency of the input is near one cycle per second. The remainder of this paper contains the analyses described.

CHAPTER III

THE SLOW SYSTEM WITH SMALL INITIAL ERROR

In this chapter, the operation of the slow system, shown in Figure 1, will be analyzed. The slow system is characterized by the

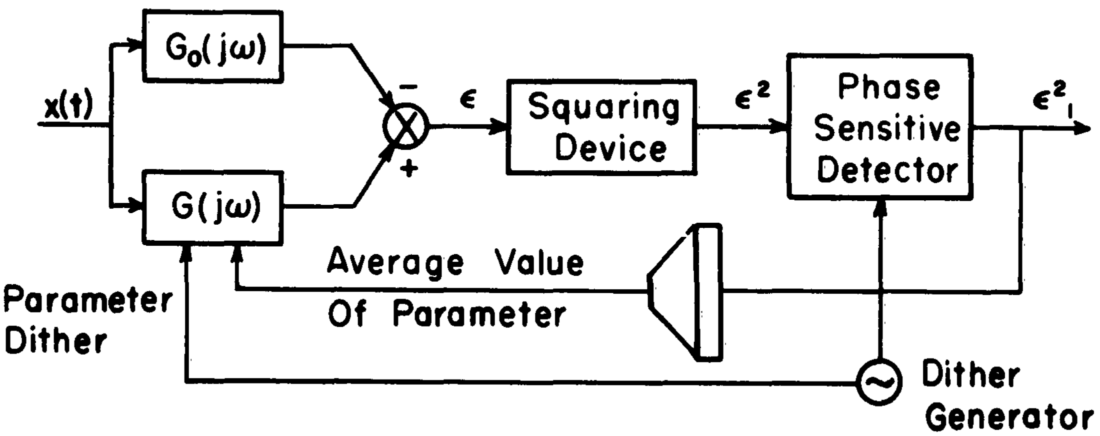

Fig. 1. Block diagram of the slow system.

fact that the parameter variation is sufficiently slow that the adjusted system is essentially in the steady state during the adaption process. The solution here is presented for the case in which the parameter a_o, the steady state gain of the adjusted system is varied. This example is selected as being representative of this type of system, although any of the system parameters can be adjusted similarly. Because of the nature of the slow system, the results here are deterministic rather than probabalistic since the solution has been derived on the basis of expected values of the signal in the adaptive loop, neglecting all statistical fluctuations.

In Figure 1, $G_o(j\omega)$ is the transfer function of the model system while $G(j\omega)$ is the transfer function of the system being adjusted. Consider the model system to have the transfer function

$$G_o(j\omega) = \frac{1}{-a_2\,\omega^2 + j\,a_1\,\omega + a_o}$$

where a_i's are constants. If an adaptive system is operated in parallel with the model system, the difference between the system outputs, ϵ, provides a signal suitable for the adaptive system's operating criterion. The transfer function of the adjusted system has the form

$$G(j\omega) = \frac{1}{-a_2\,\omega^2 + j\,a_1\,\omega + a_o + \delta(t) + b_1 \cos\omega_1 t}$$

where the a_i's are the same as those in G_o, $\delta(t)$ is a small deviation about a_o, and $b_1 \cos\omega_1 t$ is a dither signal added to the terms $a_o+\delta(t)$. The systems will be driven by x(t), a random function of time having the form[1]

$$x(t) = \sum_{[n]} a_{[n]}\, e^{jn\Delta t}$$

where the n's are irrational numbers so that one and only one exists between two adjacent integers.

The dither frequency ω_1 is much less than the mean frequency* of the input signal x(t) in the so-called "slow" system. The slow system is characterized by another condition, namely, that the dither frequency is low with respect to the average signal frequency and it may be assumed that the only signal passing through the phase sensitive detector is a d-c term corresponding to the magnitude of the error squared term at the dither frequency. The statistical type fluctuations at the output of this detector are attenuated by the low-pass filter following this detector. As shown in Figure 1, the difference in output of the two systems, ϵ, is squared and passed through the phase sensitive detector. The component of the squared signal at frequency ω_1, the dither frequency, is denoted as $\epsilon^2{}_1$. The criterion assumed for the adaptive process is that of adjusting $a_o + \delta(t)$ so as to force $\epsilon^2{}_1$ to zero. The technique for analyzing the operation of this system is represented by the block diagram in Figure 2. This technique is an expansion of the system shown in Figure 1. One approximation is introduced in that components at frequency $\omega \pm 2\omega_1$ and higher are neglected as indicated. The

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*Here mean frequency is defined in the sense

$$\overline{\omega} = \frac{\int_0^\infty \omega \, \Phi_i(\omega) d\omega}{\int_0^\infty \Phi_i(\omega) d\omega},$$ where $\Phi_i(\omega)$ is the power density spectrum of the input.

modulator blocks [2] shown in Figure 2 introduce the time-varying portion of the total signal due to the dithering of the parameter a_o. The total output is the sum of the outputs labeled* Y_1, Y_{2+}, and Y_{2-} as shown in Figure 2.

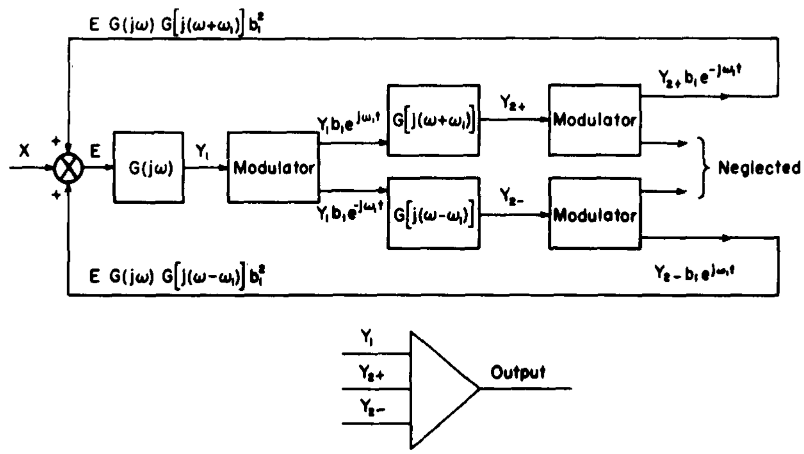

Fig. 2. Alternative representation of the system shown in Figure 1.

Assuming that the input x contains only the frequency ω, it may be represented in complex form as $X(j\omega)$. Then the adaptive system may be represented as shown in Figure 2. The modulator blocks multiply their input signals by $b_1 (e^{j\omega_1 t} + e^{-j\omega_1 t})$ where b_1 is small. Again, capital letters are used to represent complex quantities.**

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* The following subscript convention is used: a plus (+) subscript indicates that the term is a function of its regular argument $+\omega_1$. E. g., $Y_{2+} = Y_2(\omega + \omega_1)$. The negative sign means the opposite.

** The complex notation used herein is written so that the physical signal or signal component x(t) is obtained from the complex representation X of the signal as follows:

$$x(t) = X e^{j\omega t} + X^* e^{-j\omega t} \quad .$$

From Figure 2, the relation between E and X may be written by inspection.

(3-1) $$E = X + E\, b_1^2\; G(j\omega)\left[\, G(j(\omega + \omega_1)) + G(j(\omega - \omega_1))\right]$$

Rearranging terms in Eq. (3-1),

(3-2) $$E = \frac{X}{1 - b_1^2\; G\,(G_+ + G_-)}$$

Now writing Y_1, Y_{2+}, and Y_{2-} in terms of the input and the gains,

(3-3) $$Y_1 = EG = \frac{XG}{1 - b_1^2\; G\,(G_+ + G_-)}$$

(3-4) $$Y_{2+} = Y_1\, b_1\; G_+\, e^{j\omega_1 t} = \frac{b_1\; X\, GG_+\, e^{j\omega_1 t}}{1 - b_1^2\; G(G_+ + G_-)}$$

(3-5) $$Y_{2-} = Y_1\; b_1\; G_-\; e^{j\omega_1 t} = \frac{b_1\; X\, GG_-\; e^{-j\omega_1 t}}{1 - b_1^2\; G\,(G_+ + G_-)}$$

If the same type of notation is used the output of the model system may be written

(3-6) $$Y_m = X\, G_o(j\omega) = X\, G_o$$

In terms of Eqs. (3-3) through (3-6)

(3-7) $$\mathcal{E} = Y_1 + Y_{2+} + Y_{2-} + (-\,Y_m)$$

or

$$\mathcal{E} = \frac{X[G + b_1 \; GG_+ e^{j\omega_1 t} + b_1 \; GG_- e^{-j\omega_1 t} - G_o(1 - b_1{}^2 G(G_+ + G_-))]}{1 - b_1{}^2 \; G(G_+ + G_-)} \tag{3-8}$$

Now assuming that terms involving $b_1{}^2$ may be neglected,* Eq. (3-8) reduces to

$$\mathcal{E} = X\,[\,G - G_o + b_1 \; GG_+ \, e^{j\omega_1 t} + b_1 \; GG_- \, e^{-j\omega_1 t}] \tag{3-9}$$

It can be seen from Figure 1 that the signal now passes through the squaring device and then the phase sensitive detector. These two physical operations correspond mathematically to squaring ϵ and retaining only those terms at frequency ω_1. Prior to the squaring, however, the expression for ϵ will be modified to represent the signal due to the random input x, and then simplified. In terms of the random input variable x,

$$\epsilon = \sum_{[n]} a_{[n]} \, e^{jn\Delta t}[\,G(jn\Delta) - G_o(jn\Delta) + b_1 \, G(jn\Delta)G(j(n\Delta + \omega_1))\, e^{j\omega_1 t} \tag{3-10}$$

$$+ \, b_1 \; G(\,jn\Delta)\; G(j\,(n\Delta - \omega_1))\, e^{-j\omega_1 t}\,].$$

Here ϵ is the real signal.

Henceforth the brackets will be omitted from $[n]$ with the

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* This corresponds to opening the feedback loops shown in Figure 2.

understanding that when n is used as an index, it refers to the nearest integer to n.

Consider for the moment only the term $G-G_o$ from Eq. (3-10)

$$G-G_o = \frac{1}{-a_2\,\omega^2 + a_o + j a_1 \omega + \delta} - \frac{1}{-a_2\,\omega^2 + a_o + j a_1 \omega} \tag{3-11}$$

If one makes the approximation $\frac{1}{1+\delta} \cong 1 - \delta$ when $\delta \ll 1$, and since δ is a small deviation of a_o about equilibrium,

$$G-G_o \cong -\delta\, G_o^{\,2} \tag{3-12}$$

The conditions for which Eq. (3-12) is valid are described in Appendix II. Now consider the product $G(j\omega)\, G(j(\omega \pm \omega_1))$. This product is written at length as

$$G(j\omega)\, G(j(\omega \pm \omega_1)) = \frac{1}{-a_2\,\omega^2 + a_o + j a_1 \omega + \delta} \times \tag{3-13}$$

$$\times \frac{1}{-a_2\,(\omega \pm \omega_1)^2 + a_o + j\, a_1\,(\omega \pm \omega_1) + \delta}\ .$$

If one uses the same simplification as above,

$$G(j\omega)\, G(j(\omega \pm \omega_1)) \cong G_o(j\omega)(1 - \delta\, G_o(j\omega)) \times \tag{3-14}$$

$$\times\ G_o(j(\omega \pm \omega_1))(1 - \delta\, G_o(j(\omega \pm \omega_1)))$$

Now if terms in δ^2 are dropped,

$$(3\text{-}15)\quad G(j\omega)\,G(j(\omega+\omega_1)) \cong G_o(j\omega)\,G_o(j(\omega\pm\omega_1))\times$$

$$\times\,(1-\delta\,(G_o(j\omega)+G_o(j(\omega\pm\omega_1)))).$$

Substituting Eqs. (3-12) and (3-14) into Eq. (3-10) results in

$$(3\text{-}16)\quad \epsilon = \sum_n a_n e^{jn\Delta t}\Big[-\delta\,G_o^2 + b_1\,G_o\,G_{o+}\,(1-\delta\,(G_o+G_{o+}))\,e^{j\omega_1 t}$$

$$+\,b_1\,G_oG_{o-}\,(1-\delta\,(G_o+G_{o-}))\,e^{-j\omega_1 t}\Big]$$

Now Eq. (3-16) will be squared so as to retain only those terms at frequency ω_1. Furthermore, in squaring the summation, only terms of the form $a_n a_m e^{j(n+m)\Delta t}$ where $n = -m$ will be retained. Hence, in each product above, the index n will be used in the first term of each product and $-n$ with the second. Finally, all terms in δ^2 will be dropped. This part of the error squared will be labeled $\epsilon^2{}_o$.

$$(3\text{-}17)\quad \epsilon^2{}_o = \sum_n a_n e^{jn\Delta t}\left(-\delta\,G_o^2(jn\Delta)\right)\left\{\sum_{-n} a_{-n}e^{-jn\Delta t}\Big[b_1\,G_o(-jn\Delta)\right.$$

$$G_o\left(-j(n\Delta-\omega_1)\right)e^{j\omega_1 t} + b_1\,G_o(-jn\Delta)$$

$$\left.G_o\left(-j(n\Delta+\omega_1)\right)e^{-j\omega_1 t}\Big]\right\}$$

$$(3\text{-}18)\quad \epsilon^2{}_o = \sum_n a_n{}^2 \left(-\delta\,|G_o|^2\right)\left\{b_1 G_o G_{o-}^* e^{j\omega_1 t} + G_o G_{o+}^* e^{-j\omega_1 t}\right\}$$

Now if the discrete sum is replaced by an infinite integral, $\epsilon^2{}_o$ may be written as

$$(3\text{-}19)\quad \epsilon^2{}_o = -\delta b_1 \int_{-\infty}^{\infty} \Phi(\omega)\,|G_o|^2 \left\{G_o G_{o-}^* e^{j\omega_1 t} + G_o^* G_{o-} e^{-j\omega_1 t}\right\} d\omega\ .$$

$\Phi(\omega)\,d\omega$ is the continuous equivalent of the infinite sum $\Sigma\ a_n{}^2$. Here the power contained in one term $a_m{}^2$ is equal to $\int_{\omega_1}^{\omega_1+\Delta} \Phi(\omega)\,d\omega$ where the interval from ω_1 to $\omega_1+\Delta$ includes the m^{th} term, namely, $m\Delta$ in the infinite sum. If it is recalled that this signal is passed through the phase sensitive detector, the term $e^{\pm j\omega_1 t}$ may be dropped from each factor in Eq. (3-19). This represents taking the magnitude of the in-phase term, and will be indicated by the subscript 1 on ϵ^2. A description of the operation of the phase sensitive detector is given in Appendix I.

Note that the term in the brackets in Eq. (3-19) is real, after the $e^{\pm j\omega_1 t}$ terms are dropped. Furthermore, it is equal to twice the real part of the first term. If one uses this, as well as further simplifications, the integral in Eq. (3-19) can be reduced.

$$(3\text{-}20)\quad 2\,\mathrm{Re}\,(G_o G_{o-}^*) = 2\,\mathrm{Re}\left\{\frac{1}{-a_2\omega^2+a_o+ja_1\omega}\cdot\frac{1}{-a_2(\omega-\omega_1)^2+a_o+ja_1(\omega-\omega_1)}\right\}.$$

Noting that $\omega_1 << \omega$, the above expression may be written as

$$(3\text{-}21)\quad 2\ \mathrm{Re}\ (G_o G_{o-}^{\ *}) = 2\ \mathrm{Re}\left\{ G_o \frac{1}{-a_2\,\omega^2 + a_o - j\,a_1\,\omega + D} \right\}$$

where $D = 2\ a_2\ \omega\omega_1\ - a_2\,\omega_1^{\ 2}\ + j\ a_1\,\omega_1$. Thus

$$(3\text{-}22)\quad 2\ \mathrm{Re}\ (G_o G_{o-}^{\ *}) = 2\ \mathrm{Re}\left\{ G_o G_o^{\ *}\ (1 - D\ G_o^{\ *}) \right\}$$

$$= 2\ \mathrm{Re}\left\{ G_o G_o^{\ *}\left[1 - (2a_2\ \omega\omega_1\ - a_2\,\omega_1^{\ 2} + j\ a_1\,\omega_1)\ G_o^{\ *}\right]\right\} .$$

Consider the following:

$$(3\text{-}23)\quad \mathrm{Re}\ \left[1 - (2a_2\ \omega\omega_1\ - a_2\,\omega_1^{\ 2} + j\ a_1\,\omega_1)\ G_o^{\ *}\right]$$

$$= \mathrm{Re}\left[1 - \frac{2\ a_2\,\omega\omega_1\ - a_2\,\omega_1^{\ 2}\ + j\ a_1\,\omega_1}{-\ a_2\ \ \omega^2\ + a_o\ j\ a_1\ \omega}\right]$$

$$= \left[1 - \frac{(2\ a_2\ \omega\omega_1\ - a_2\ \omega_1^{\ 2}\,)\ (-a_2\,\omega^2 + a_o) - (a_1^{\ 2}\,\omega_1\,\omega)}{(-\ a_2\,\omega^2\ + a_o)^2 + (a_1\,\omega)^2}\right]$$

Since the integral has limits $-\infty$ to ∞, the terms in odd powers of ω contribute nothing; Thus Eq. (3-19) becomes

$$(3\text{-}24)\quad \epsilon^2{}_1 \doteq -\ \delta b_1 \int_{-\infty}^{\infty} \Phi(\omega)\ |G_o|^4 \left[1 + \frac{a_2\,\omega_1^{\ 2}\ (-a_2\,\omega^2 + a_o)}{(-a_2\,\omega^2 + a_o)^2 + (a_1\,\omega)^2}\right] d\omega\ .$$

With the assumption that $\Phi(\omega) = \dfrac{1}{\omega^2 + \beta^2}$, the integral can be evaluated.

$$(3\text{-}25)\quad \epsilon^2{}_1 = -\delta b_1 \int_{-\infty}^{\infty} \left(\frac{1}{\beta + j\omega}\ \frac{1}{\beta - j\omega}\right)\left[\frac{1}{-a_2\,\omega^2 + a_o + ja_1\omega}\ \frac{1}{a_2\,\omega^2 + a_o + ja_1\,\omega}\right]^2$$

$$\left[1 + \frac{-a_2{}^2\,\omega_1{}^2\,\omega^2 \quad + \quad a_o\, a_2\ \omega_1{}^2}{(-a_2\, a^2 + a_o + j\,a_1\,\omega)\ (-a_2\,\omega^2 + a_o - j\ a_1\ \omega\ )}\right] d\omega$$

Using the integral table in Reference 3, the above expression reduces to

$$(3\text{-}26)\qquad \epsilon^2{}_1 = -\delta b_1\ (I_5 + I_7\ )\ ;$$

and if terms involving $\omega_1{}^2$ are neglected,

$$(3\text{-}27)\qquad \epsilon^2{}_1 = -\delta\, b_1\ I_5\ .$$

This may be interpreted as follows: The adaptive system will tend to reduce $\epsilon^2{}_1$ to zero; when this is accomplished, δ above must be zero; thus the adaptive system becomes identical with the model system.

The next question which arises is that of determining the form of $\epsilon^2{}_1\ (\ \delta\ (t)\ )$ in the case where some or all of the foregoing approximations are not made.

As can be seen from Eq. (3-16) and those equations immediately preceding, if no approximations are made initially, the final form of the term $\epsilon^2{}_1$ will be

$$\epsilon^2{}_1 = C_1\ \delta + C_2\ \delta^2 + \ldots\ldots.$$

where the C_i's are constants and $C_{i+i} << C_i$. Suppose $C_3 = C_4 = \ldots = C_n = 0$. Then $\epsilon^2{}_1 = C_1 \delta + C_2 \delta^2$.

Setting $\epsilon^2{}_1 = 0$ gives

$$\delta = 0 \text{ and } \delta = -\frac{C_1}{C_2}$$

Since $C_2 << C_1$, this implies that the error in a_o is very large. But this cannot be, since the previous analysis depends upon δ being small. Furthermore, if the adaptive system is stable in the vicinity of equilibrium or $\delta = 0$, any deviation from zero will tend to return to zero. Once the expression for $\epsilon^2{}_1$ is in the form of Eq. (3-27), it is possible to conduct a stability analysis for the system by using classical nonlinear methods, such as the small signal theory for the system near equilibrium or other applicable methods. The behavior of the system for large initial values of δ (t) will be considered in the next chapter.

CHAPTER IV

THE SLOW SYSTEM WITH LARGE INITIAL ERROR

The previous analysis of the slow system has been done with the assumption that the initial value of $\delta(t)$ is small enough for terms in δ^2 to be neglected. As indicated at the end of Chapter III, the system should be studied for large values of δ, or values away from equilibrium in order to determine the conditions for instability, if any exist. The error, $\mathcal{E}$ before any approximations are made, can be written as Eq. (3-8) in complex form.

$$(4\text{-}1) \quad \mathcal{E} = \frac{X\,[\,G + b_1\ GG_+ e^{j\omega_1 t} + b_1\ GG_- e^{-j\omega_1 t} - G_o\,(1 + b_1{}^2\, G(G_+ + G_-))\,]}{1 - b_1{}^2\, G\,(G_+ + G_-)}$$

Again, assuming that b_1 is $\ll 1$ and $b_1\,|G|$ in all cases is smaller than unity, the following approximation can be made.

$$(4\text{-}2) \quad \mathcal{E} = X\ [\,G + b_1\ GG_+ e^{j\omega_1 t} + b_1\ GG_-\, e^{-j\omega_1 t} - G_o\,]$$

At this point it is desirable to assume values for the constants in the gain expressions and evaluate the error ϵ. Inserting the summation expression for $x(t)$, and representing $(n\,\Delta)$ in the gain expressions by ω, ϵ becomes

$$(4\text{-}3) \quad \epsilon = \sum_{[n]} a_{[n]}\, e^{jn\Delta t} \Bigg\{ \frac{1}{-a_2\,\omega^2 + a_o + ja_1\,\omega + \delta} - \frac{1}{-a_2\,\omega^2 + a_o + ja_1\,\omega}$$

$$+ b_1 \left[\frac{1}{-a_2\,\omega^2 + a_o + ja_1\,\omega + \delta} \cdot \frac{1}{-a_2\,(\omega + \omega_1)^2 + a_o + ja_1\,(\omega + \omega_1) + \delta} \right] e^{j\omega_1 t}$$

$$+ b_1 \left[\frac{1}{-a_2\,\omega^2 + a_o + ja_1\,\omega + \delta} \cdot \frac{1}{-a_2\,(\omega - \omega_1)^2 + a_o + ja_1\,(\omega - \omega_1) + \delta} \right] e^{-j\omega_1 t} \Bigg\}$$

The brackets on the index $[n]$ will again be omitted. At this point, let $a_2 = a_1 = a_o = 1$.

$$(4\text{-}4) \quad \epsilon = \sum_{n} a_n\, e^{jn\Delta t} \Bigg[\frac{1}{-\omega^2 + 1 + j\omega + \delta} - \frac{1}{-\omega^2 + 1 + j\omega}$$

$$+ b_1 \frac{1}{-\omega^2 + 1 + j\omega + \delta} \cdot \frac{1}{-(\omega + \omega_1)^2 + 1 + j\,(\omega + \omega_1) + \delta}\, e^{j\omega_1 t}$$

$$+ b_1 \frac{1}{-\omega^2 + 1 + j\omega + \delta} \cdot \frac{1}{-(\omega - \omega_1)^2 + 1 + j\,(\omega - \omega_1) + \delta}\, e^{-j\omega_1 t} \Bigg]$$

To simplify the analysis, it is expedient to square ϵ at this time, neglecting all terms at frequencies other than ω_1. Note that in the squaring operation, the right-hand member of each product will appear as complex conjugates as in Chapter III. Again, this term will be denoted by the subscript 1.

$$(4\text{-}5) \qquad \epsilon^2{}_1 = \sum_n a_n{}^2 \left\{ \left(\frac{1}{-\omega^2 + 1 + \delta + j\omega} - \frac{1}{-\omega^2 + 1 + j\omega} \right) \right.$$

$$\left[\left(b_1 \frac{1}{-\omega^2 + 1 + j\omega} \cdot \frac{1}{-(\omega + \omega_1)^2 + 1 + \delta - j(\omega + \omega_1)} \right) \right.$$

$$\left. \left. + \left(b_1 \frac{1}{-\omega^2 + 1 + \delta - j\omega} \quad \frac{1}{-(-\omega + \omega_1)^2 + 1 + \delta - j(\omega - \omega_1)} \right) \right] \right\}$$

Note that the $e^{\pm j\omega_1 t}$ terms have been omitted; this is a result of passing these terms through the phase sensitive detector (see Appendix I). Now from this

$$(4\text{-}6) \qquad \epsilon^2{}_1 = \sum_n a_n{}^2 \left[\frac{1}{(-\omega^2 + 1 + \delta + j\omega)(-\omega^2 + 1 + j\omega)} \right]$$

$$\left\{ \frac{b_1}{(-\omega^2 + 1 + \delta - j\omega)(-(\omega + \omega_1)^2 + 1 + \delta - j(\omega + \omega_1))} \right.$$

$$\left. + \frac{b_1}{(-\omega^2 + 1 + \delta - j\omega)(-(\omega - \omega_1)^2 + 1 + \delta - j(\omega - \omega_1))} \right\}$$

$$(4\text{-}7) \quad \epsilon^2{}_1 = \sum_n a_n^2 \left\{ \frac{-\delta b_1}{|-\omega^2+1+\delta+j\omega|^2(-\omega^2+1+j\omega)(-(\omega+\omega_1)^2+1+\delta-j(\omega+\omega_1))} \right.$$

$$\left. + \frac{-\delta b_1}{|-\omega^2+1+\delta+j\omega|^2(-\omega^2+1+j\omega)(-(\omega-\omega_1)^2+1+\delta-j(\omega-\omega_1))} \right\}$$

$$(4\text{-}8) \quad \epsilon^2_1 = \sum_n a_n^2 \quad \frac{(-\delta b_1)}{|-\omega^2+1+\delta-j\omega|^2}$$

$$\left\{ \frac{1}{(-\omega^2+1+j\omega)(-(\omega+\omega_1)^2+1+\delta-j(\omega+\omega_1))} \right.$$

$$\left. + \frac{1}{(-\omega^2+1+j\omega)(-(\omega-\omega_1)^2+1+\delta-j(\omega-\omega_1))} \right\}$$

It will be assumed in the following equations that ω_1 is sufficiently small for $\omega_1{}^2$ terms to be neglected.

Expanding the fractions in brackets and combining gives

$$\left\{ \frac{1}{(-\omega^2+1+j\omega)(-\omega^2-2\,\omega\omega_1+1+\delta-j\omega-j\omega_1)} \right.$$

$$\left. + \frac{1}{(-\omega^2+1+j\omega)(-\omega^2+2\,\omega\omega_1+1+\delta-j\omega+j\omega_1)} \right\}$$

Factoring out $\dfrac{1}{(-\omega^2 + 1 - j\omega)}$,

$$\text{(4-9)} \quad \text{Brackets} = \left\{ \frac{1}{|-\omega^2 + 1 + j\omega|^2 \left(1 + \dfrac{\delta - 2\,\omega\omega_1 - j\omega_1}{-\omega^2 + 1 - j\omega}\right)} + \frac{1}{|-\omega^2 + 1 + j\omega|^2 \left(1 + \dfrac{\delta + 2\,\omega\omega_1 + j\omega_1}{-\omega^2 + 1 - j\omega}\right)} \right\}$$

Upon rewriting $\epsilon^2{}_1$, one can write Eq. (4-9) as

$$\text{(4-10)} \quad \epsilon^2{}_1 = \sum_n a_n{}^2 \frac{-\delta b_1}{|-\omega^2 + 1 + j\omega|^4} \left\{ \frac{1}{1 + \dfrac{\delta - 2\,\omega\omega_1 - j\,\omega_1}{-\omega^2 + 1 - j\omega}} + \frac{1}{1 + \dfrac{\delta + 2\,\omega\omega_1 + j\,\omega_1}{-\omega^2 + 1 - j\omega}} \right\} .$$

If the fractions in the brackets are rationalized,

$$\left\{ \frac{-\omega^2 + 1 - j\omega}{-\omega^2 + 1 - j\omega + \delta - 2\,\omega\omega_1 - j\omega_1} + \frac{-\omega^2 + 1 - j\omega}{-\omega^2 + 1 - j\omega + \delta + 2\,\omega\omega_1 + j\omega_1} \right\}$$

$$= \left\{ \frac{(-\omega^2 + 1 - j\omega)\,(-\omega^2 + 1 - j\omega + \delta - \omega^2 + 1 - j\omega + \delta)}{(-\omega^2 + 1 - j\omega + \delta - 2\,\omega\omega_1 - j\omega_1)(-\omega^2 + 1 - j\omega + \delta + 2\,\omega\omega_1 + j\omega_1)} \right\}$$

$$= \left\{ \frac{(-\omega^2 + 1 - j\omega)\; 2\,(-\omega^2 + 1 - j\omega + \delta)}{(-\omega^2 + 1 - j\omega + \delta)^2 - (2\,\omega\omega_1 + j\omega_1)^2} \right\}$$

Therefore

$$(4\text{-}11)\quad \epsilon^2{}_1 = \sum_n a_n^{\,2}\ \frac{-\delta b_1}{|-\omega^2+1+j\omega|^4}\left\{\frac{(-\omega^2+1-j\omega)2(-\omega^2+1-j\omega+\delta)}{(-\omega^2+1-j\omega+\delta)^2-(2\omega\omega_1+j\omega_1)^2}\right\}$$

If $\omega_1 << \omega_{average}$, the terms in $\omega_1{}^2$ can be neglected with respect to the others in the denominator of the expression in brackets,

$$(4\text{-}12)\quad \epsilon^2{}_1 = \sum_n a_n^{\,2}\ \frac{-2\delta b_1}{|-\omega^2+1+j\omega|^4}\left(\frac{-\omega^2+1-j\omega}{-\omega^2+1-j\omega+\delta}\right).$$

Note that this reduces to the expression derived in the previous chapter when the approximation of δ small is used if the approximation of neglecting $\omega_1{}^2$ is applied to Eq. (3-20) and those that follow.

Replacing the discrete summation by the equivalent continuous form, Eq. (4-12) may be written

$$(4\text{-}13)\quad \epsilon^2{}_1 = -2\delta b_1\int_{-\infty}^{\infty}\Phi(\omega)\frac{1}{|-\omega^2+1+j\omega|^4}\left(\frac{-\omega^2+1-j\omega}{-\omega^2+1-j\omega+\delta}\right)d\omega\ .$$

This may also be evaluated by using the technique mentioned in the previous chapter. In this case, however, the δ term appears in the denominator and it must be expanded in order to see how the function behaves in terms of δ. First, one must rationalize the expression in Eq. (4-13) above.

$$(4\text{-}14)\quad \epsilon^2{}_1 = -2\,\delta\, b_1 \int_{-\infty}^{\infty} \frac{\Phi(\omega)}{|-\omega^2 + 1 + j\omega|^4}\;\frac{(-\omega^2 + 1 - j\omega)(-\omega^2 + 1 + \delta - j\omega)}{(-\omega^2 + 1 + \delta)^2 + \omega^2}$$

$$(4\text{-}15)\quad \epsilon^2{}_1 = -2\,\delta b_1 \int_{-\infty}^{\infty} \frac{1}{\beta + j\omega} \cdot \frac{1}{\beta - j\omega} \cdot \frac{1}{(-\omega^2 + 1 + j\omega)^2} \cdot \frac{1}{(-\omega^2 + 1 - j\omega)^2}$$

$$\cdot \frac{\omega^4 - (2+\delta)\,\omega^2 - \delta\, j\omega + 1 + \delta}{(-\omega^2 + 1 + \delta + j\omega)(-\omega^2 + 1 + \delta - j\omega)}\quad d\omega$$

The format of the integral as given in Reference 3 is

$$(4\text{-}16)\qquad I_n = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left| \frac{c_{n-1}\, s^{n-1} + \ldots + c_o}{d_n\, s^n + \ldots + d_o} \right|^2 ds\ .$$

The terms in the expression found in Reference 3 are equated to the terms of Eq. (4-15) as shown in Table I.

TABLE I

EVALUATION OF TERMS

$d_7 = 1$
$d_6 = +(3 + \beta)$
$d_5 = +(6 + 3\beta + \delta)$
$d_4 = (7 + 6\beta + 2\delta + \beta\delta)$
$d_3 = +(6 + 7\beta + 3\delta + 2\beta\delta)$
$d_2 = (3 + 6\beta + 2\delta + 3\beta\delta)$
$d_1 = (1 + 3\beta + \delta + 2\beta\delta)$
$d_o = \beta\,(1 + \delta)$

$2c_4\, c_o - 2c_3\, c_1 + c_2{}^2 = 1$
$2c_2\, c_o - c_1{}^2 = 2 + \delta$
$c_o{}^2 = 1 + \delta$

The important feature is not the exact expression for $\epsilon^2{}_1$ but rather the form of the expression in terms of δ.

The data in Table 2 show the highest and lowest power of δ in each term of the evaluation in I_7.

TABLE 2

POWERS OF δ IN NUMERATOR AND DENOMINATOR

Term	Highest power of δ	Lowest power of δ
m_o	5	0
m_1	4	0
m_2	4	0
m_3	4	0
m_4	4	0
m_5	4	0
m_6	4	0
$c_o^2\, m_6$	5	0
$(c_1^2 - 2c_1 c_3 + 2c_o c_4)\, m_4$	5	0
$\Delta\, 7$	6	0

Then, the form of $\epsilon^2{}_1$ becomes

$$(4\text{-}17) \qquad \epsilon^2{}_1 = \frac{b_1\delta\left[k_o + k_1\,\delta + k_2\,\delta^2 + k_3\,\delta^3 + k_4\,\delta^4 + k_5\delta^5\right]}{c_o + c_1\,\delta + c_2\,\delta^2 + c_3\,\delta^3 + c_4\,\delta^4 + c_5\,\delta^5 + c_6\,\delta^6}$$

where the c_i's and k_i's are constants. Thus, for $\delta << 1$, as in Chapter III, $\epsilon^2{}_1$ becomes a linear function of δ; while for δ large, so that $\delta^5 >> \delta^4$, $\epsilon^2{}_1$ becomes essentially independent of δ. Again, as in Chapter III, this form for ϵ^2_1 is suited to doing a stability analysis using conventional nonlinear techniques.

CHAPTER V

THE FAST SYSTEM WITH SMALL INITIAL ERROR

One of the goals in every analysis of an adaptive control system is the determination of the variation of the adjusted parameter about its optimum value. It is tempting in the study of these variations simply to approximate the spectrum at the output of the criterion detector on the basis that the adaptive or supervisory loop is opened and on the assumption that no parameter variations occur. This assumption, although it is often valid as the first step of an iterative determination of nonlinear system characteristics, often leads to a contradiction in adaptive system analysis which must be resolved. This contradiction arises as follows: it is assumed that the adaptive process adjusts itself so that the average value of the parameter error is zero or at least near zero. Yet the spectrum of the fluctuation of this parameter about its average value (called hunting loss), if calculated on the basis of the assumption just outlined, may well be infinite at zero frequency indicating not only a large and very slow variation of the parameter about its average value, but also the possibility of a large average deviation from the desired value. This apparent contradiction cannot be resolved by successive iterations and

it must be concluded either that the system does not adapt, or that the methodology employed has limitations.*

The latter conclusion is the case. In this chapter, the assumptions and constraints which defined the slow system will be dropped and a technique will be developed which will allow the determination of the spectrum of the signals in the supervisory section of the adaptive system under consideration. The same technique leads to a representation which can be used to determine the stability of the adaptive system.

In this chapter, an analysis of the fast system will be made. The input to the system is again the random function of time given by

$$x(t) = \sum_{[n]} a_{[n]} e^{jn\Delta t} \tag{5-1}$$

where n is almost but not quite an integer and $[n]$** is the nearest integer. The block diagram of this system is shown in Figure 3. The analysis will be carried out on the basis of observing only one frequency component of the total signal as it traverses the system. If the

---

* If the transfer function of the adaptive loop does not have a pole or poles on the imaginary axis, the iterative technique mentioned above may be satisfactory.

** As in earlier chapters, the brackets on the indices will be omitted.

expression for one component can be developed, then the expression for the total signal may be found by generalizing the procedure for all frequencies present. Before the mathematical analysis of this system is made, a description of the method of analysis will be given in order to justify this approach to the problem.

In this analysis, the system is assumed to be at equilibrium so that the integrator output has an average value of zero. Note that the model system and the adjusted system are shown as identical, except that the adjusted system has a separate feedback path through the multiplier (Figure 3). This is equivalent to stating that the constant a_o is being dithered, and is at its equilibrium value, as previously mentioned.

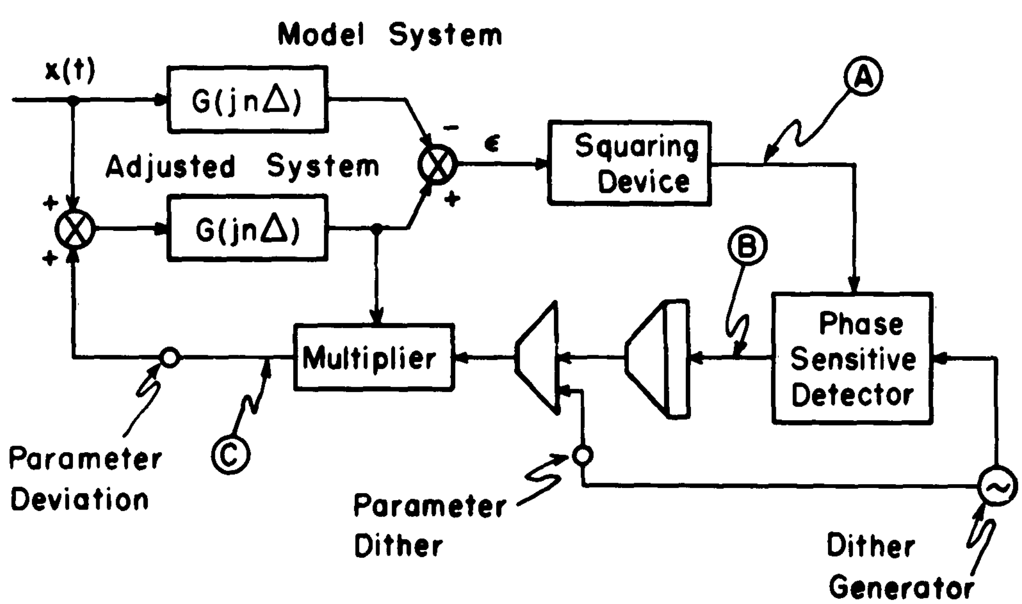

Fig. 3. Block diagram of the fast system.

Consider now a portion of the total signal appearing at point A. Let this portion of the total signal be at frequency ω_a. As this signal passes through the system, it will produce a correction term at point A which will be at the same frequency, ω_a.

This correction term, produced by the original signal at ω_a, is the only signal in the system which is coherent with the original ω_a term.* The analysis will be carried out on the basis that the total output of the squaring device can be represented as shown in Figure 4.

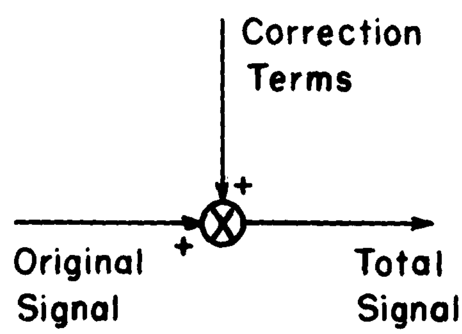

Fig. 4. Block diagram of the analysis method.

Here the one input to the differential is that part of the signal which would exist if the loop were opened at B (Figure 3). The second input is the correction term produced by the original component after the loop is closed. Thus the output is the total signal at frequency ω_a.

If the phase sensitive detector is opened at point B (Figure 3),

---

* There are in fact other signals present. However, those signals are not at the frequency ω_a and do not add coherently and are therefore neglected in the analysis to follow.

then the only signal at point C is that due to the product of the dither and the output of the adjusted system. Under this condition, the circuit can be opened at point C and a generator inserted having an output

$$E_{gen} = b_1 \sum_n C_n \left(e^{j(n\Delta+\omega_1)t} + e^{j(n\Delta-\omega_1)t}\right) \tag{5-2}$$

where b_1 is the amplitude of the dither and the summation $\sum_n C_n e^{jn\Delta t}$ represents the output of the adjusted system. Note that this is the same approximation which was used in Chapter III when terms involving b_1^2 were dropped. Since b_1 will be assumed to be small enough so that b_1^2 terms may be neglected, this is equivalent to restricting the above signal to one traversal of the adjusted system-multiplier loop. This assumption will be preserved with the circuit closed at point B.

Consider a signal at point A having the form

$$"a" = a_1 \, e^{j\omega a_1 t} \, . \tag{5-3}$$

(The notation "a" will be used in tracing this signal through the system.) At the output of the phase sensitive detector,

$$"a" = a_1 \left(e^{j(\omega a_1+\omega_1)t} + e^{j(\omega a_1-\omega_1)t}\right). \tag{5-4}$$

At the output of the integrator,

$$"a" = a_1 \left(\frac{e^{j(\omega a_1+\omega_1)t}}{j(\omega a_1+\omega_1)} + \frac{e^{j(\omega a_1-\omega_1)t}}{j(\omega a_1-\omega_1)}\right) \tag{5-5}$$

At the output of the multiplier, this signal is equal to

$$\text{(5-6)} \qquad \text{"a"} = \Sigma\, C_n\, e^{jn\Delta t}\left[b_1\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right)\right.$$

$$\left. - a_1\left(\frac{e^{j(\omega_{a_1} + \omega_1)t}}{j(\omega_{a_1} + \omega_1)} + \frac{e^{j(\omega_{a_1} - \omega_1)t}}{j(\omega_{a_1} - \omega_1)}\right)\right]$$

Here $\Sigma\, C_n e^{jn\Delta t}$ is the representation for the output signal of the adjusted system (Figure 3).

Now as this signal combines with the input and is passed through the adjusted system block, each term is multiplied by a gain at the appropriate frequency. Furthermore, these are the only terms which can contribute coherently at the output of the detector to the original signal at frequency ω_{a_1}. The sign change in Eq. (5-6) is a result of the amplifier and necessary for system stability.

The output of the differential (difference in outputs of the two systems) contains only the ω_a terms since terms due to the input signal and at input signal frequencies exactly cancel at equilibrium. Thus the signal may be written

$$(5\text{-}7) \qquad "a" = b_1 \sum_n C_n\, G\left(j(n\Delta + \omega_1)\right) e^{j(n\Delta + \omega_1)t}$$

$$+ b_1 \sum_n C_n\, G\left(j(n\Delta - \omega_1)\right) e^{j(n\Delta - \omega_1)t}$$

$$- \sum_n \frac{C_n\, a_1\, G\left(j(n\Delta + \omega_{a_1} + \omega_1)\right) e^{j(n\Delta + \omega_1 + \omega_{a_1})t}}{j(\omega_1 + \omega_{a_1})}$$

$$- \sum_n \frac{C_n\, a_1\, G\left(j(n\Delta + \omega_{a_1} - \omega_1)\right) e^{j(n\Delta + \omega_{a_1} - \omega_1)t}}{j(\omega_{a_1} - \omega_1)}$$

Note that the latter two summations represent only that portion of ϵ due to a single term in the output of the square law detector .. Finally consider the signal as it is passed through the squaring device. Here only those terms which combine and produce a signal at frequency ω_a will be retained.

To find these terms, it is necessary in each of the products that if index +n is used on one term of the product, (-n) must be used on the other. Therefore

$$(5\text{-}8) \qquad "a" = - \sum_n \left[C_{-n}\, G(-jn\Delta - j\omega_1)\right] \frac{a_1\, G(j(n\Delta + \omega_1 + \omega_{a_1}))\, C_n b_1\, e^{j\omega_{a_1} t}}{j(\omega_{a_1} + \omega_1)}$$

$$- \sum_n \left[C_{-n}\, G(-jn\Delta + j\omega_1)\right] \frac{a_1\, G(j(n\Delta - \omega_1 + \omega_{a_1}))\, C_n b_1\, e^{j\omega_{a_1} t}}{j(\omega_{a_1} - \omega_1)}$$

Finally, since the above expression represents the correction terms, and all terms are at frequency ω_{a_1}, the magnitudes of these terms may be equated giving

$$(5\text{-}9) \quad a_{10} - b_1 a_1 \left\{ \sum_n \frac{\left[C_{-n}\, G(-jn\Delta - j\omega_1)\, G(j(n\Delta + \omega_1 + \omega_{a_1}))\, C_n\right]}{j(\omega_{a_1} + \omega_1)} + \sum_n \frac{\left[C_{-n}\, G(-jn\Delta + j\omega_1)\, G(j(n\Delta - \omega_1 + \omega_{a_1}))\, C_n\right]}{j(\omega_{a_1} - \omega_1)} \right\} = a_1$$

where a_{10} is the uncorrected magnitude of the original signal and a_1 the first order approximation to the total magnitude. Now the magnitude of a_1 may be found, assuming that the expression in the brackets above is a transfer function and solving for a_1 from Eq. (5-9)

$$(5\text{-}10) \quad a_1 = a_{10} \left\{ 1 + b_1 \left[\frac{\sum_n C_n^2\, G(-jn\Delta - j\omega_1) G(j(n\Delta + \omega_1 + \omega_{a_1}))}{j(\omega_{a_1} + \omega_1)} + \frac{\sum_n C_n^2\, G(-jn\Delta + j\omega_1) G(j(n\Delta - \omega_1 + \omega_{a_1}))}{j(\omega_{a_1} - \omega_1)} \right] \right\}^{-1} .$$

Furthermore, stability of the system may be studied by considering the homogeneous equation

$$(5\text{-}11) \quad a_1 \left(1 + b_1 \left\{ \frac{\sum_n C_n^2\, G(-jn\Delta - j\omega_1)\, G(j(n\Delta + \omega_1 + \omega_{a_1}))}{j(\omega_{a_1} + \omega_1)} + \frac{\sum_n C_n^2\, G(-jn\Delta + j\omega_1)\, G(j(n\Delta - \omega_1 + \omega_{a_1}))}{j(\omega_{a_1} - \omega_1)} \right\} \right) = 0$$

and making an equivalent Nyquist plot for this system. In general, once the analysis of the system has been carried to the point of Eq. (5-9), conventional techniques may be used to analyze system behavior. The fact that an infinite summation occurs and the implication that a digital computer analysis must, in general, be used, does not alter the fact that the conventional techniques are applicable. Furthermore these summations can be converted to integral form and in theory these integrals can be evaluated exactly. However, the evaluation is extremely time-consuming and tedious.

Finally, it should be pointed out that if noise is present in the system, it can be handeled in precisely the same manner; that is, one must trace each component as it traverses the system and then sum these to find their total effect. If it is desired to find the spectrum of the output of the squaring device, this too can be approximated by conventional techniques applied to the signal as expressed in Eq. (5-10), and iterations based upon this same technique will improve the accuracy.

CHAPTER VI

ANALOG COMPUTER SOLUTION

This chapter is the description of an analog computer solution of the problem considered in Chapters III and IV. Figure 5 is an analog computer diagram of the system under observation. The variables are indicated on this computer diagram. To augment the conclusions reached in ChaptersIII and IV, four separate types of runs were made with this system. First, the system was allowed to adapt for the case that $\delta(t)$ at $t = 0$ was a small positive constant. The second run was for the case $\delta(t)$ at $t = 0$ equal to a small negative constant. Figures 6 and 7 show typical response curves for these two cases. The two curves shown on each figure are ϵ^2 and $\delta(t)$ versus cycles of the dither frequency.

Figures 6 and 7 show the manner in which $\delta(t)$ is adjusted (in this case, to zero) when its initial value is small. The time scale here has been compressed so that the response of the adaptive system may look to be faster than it really is. The parameters used for this case were $a_o = a_1 = a_2 = 1$, $\omega_1 = 0.2$ radians per second, and $x(t)$ a random signal with average frequency of about 1 cycle per second.

Figure 8 (a) shows the results in the case when the same system as used for Figures 6 and 7 is allowed to adjust with a $\delta(t)$ at $t = 0$ equal to nearly unity (100 volts is one machine unit on the analog

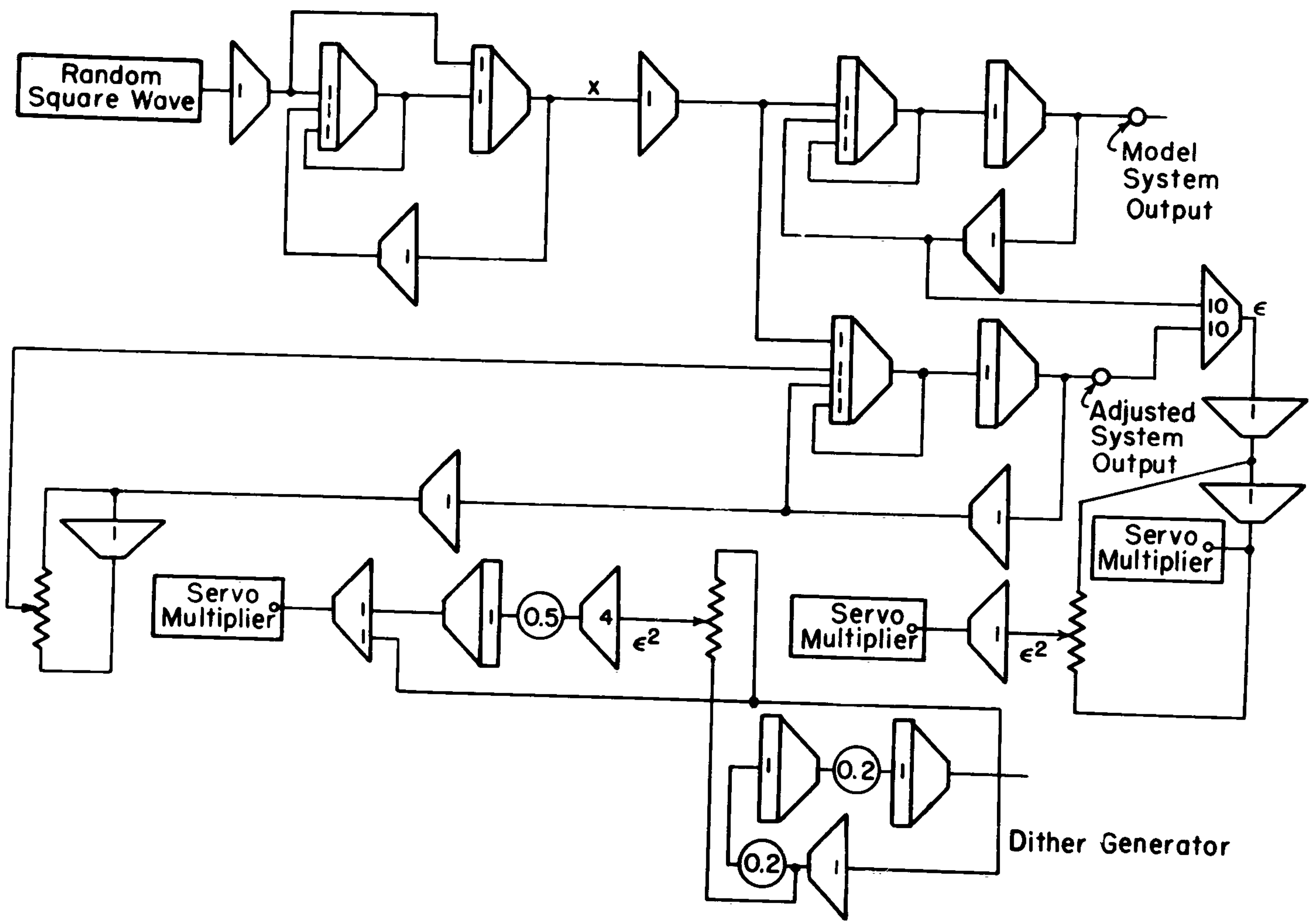

Fig. 5. Analog computer block diagram of slow system.

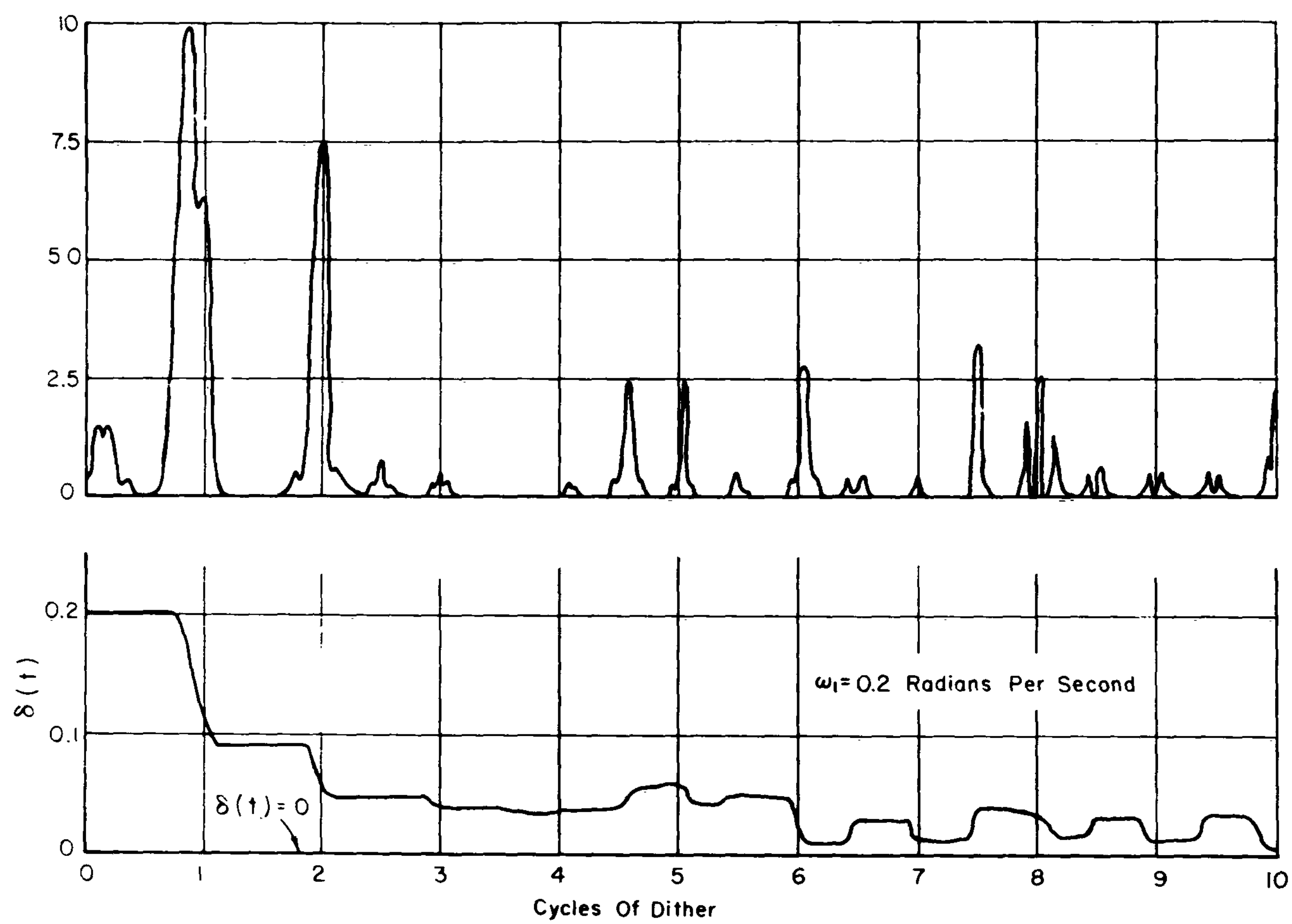

Fig. 6. ϵ^2 and $\delta(t)$ versus cycles of dither.

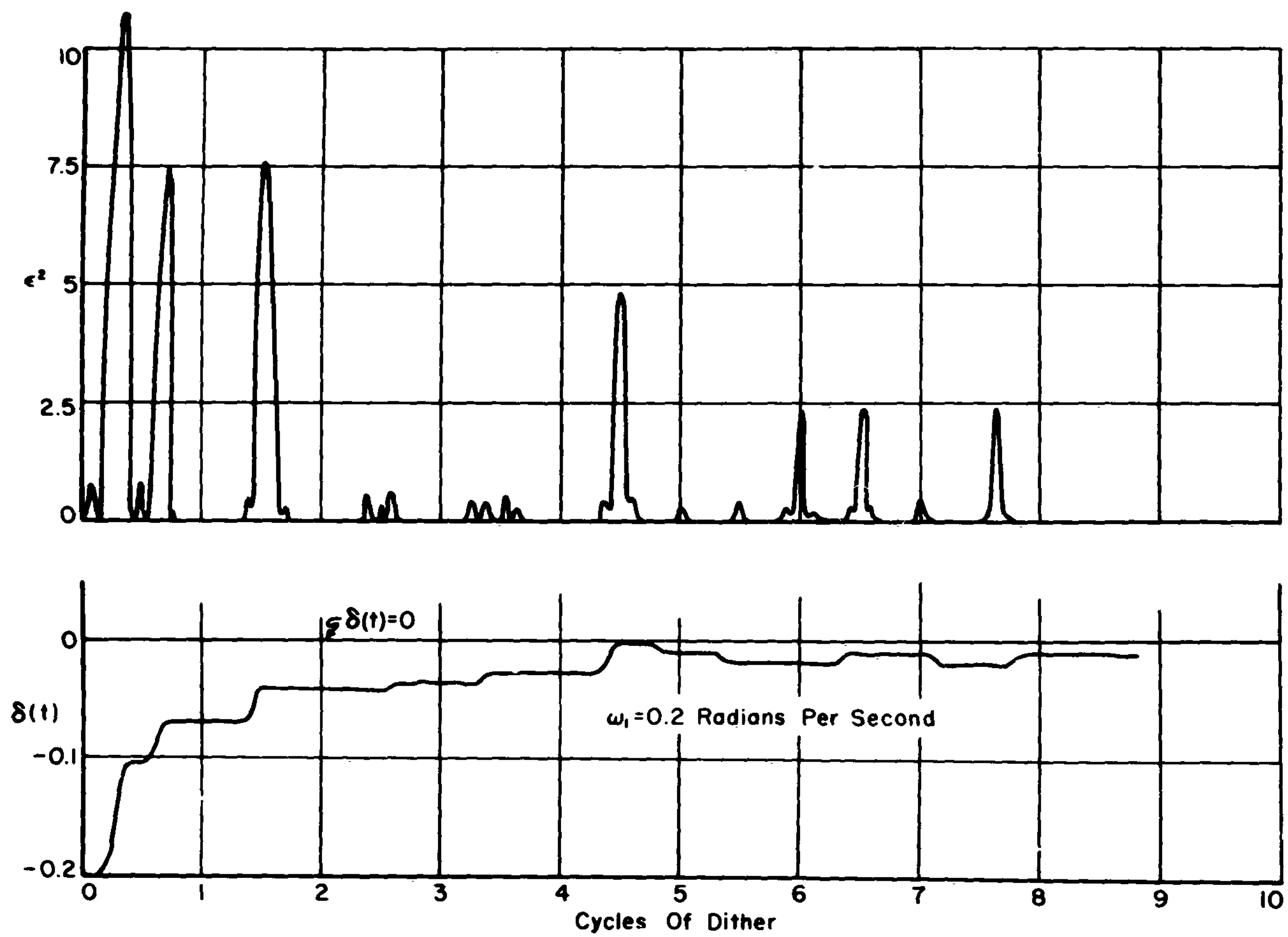

Fig. 7. ϵ^2 and $\delta(t)$ versus cycles of dither.

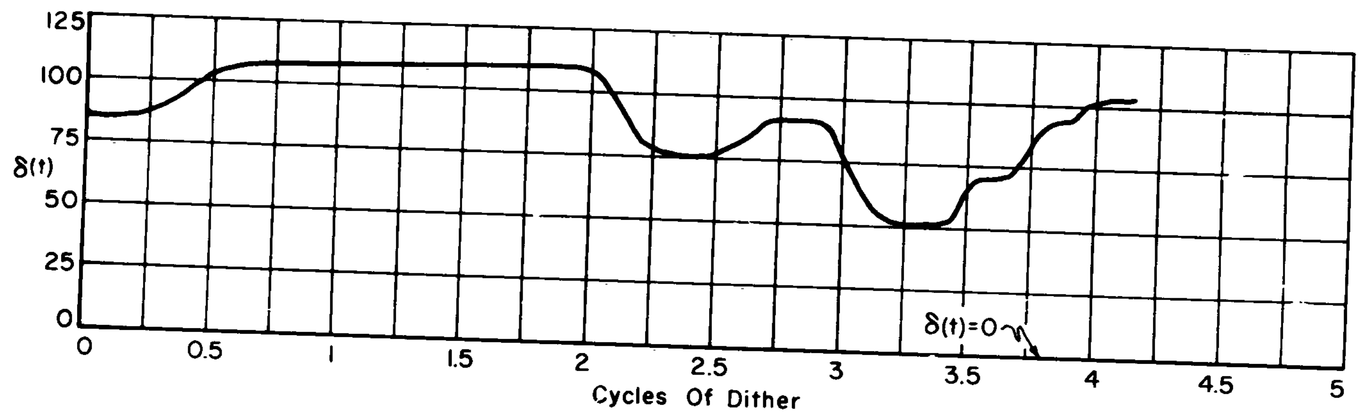

Fig. 8. (a) System of Figure 6 with δ (t) at t = 0 large

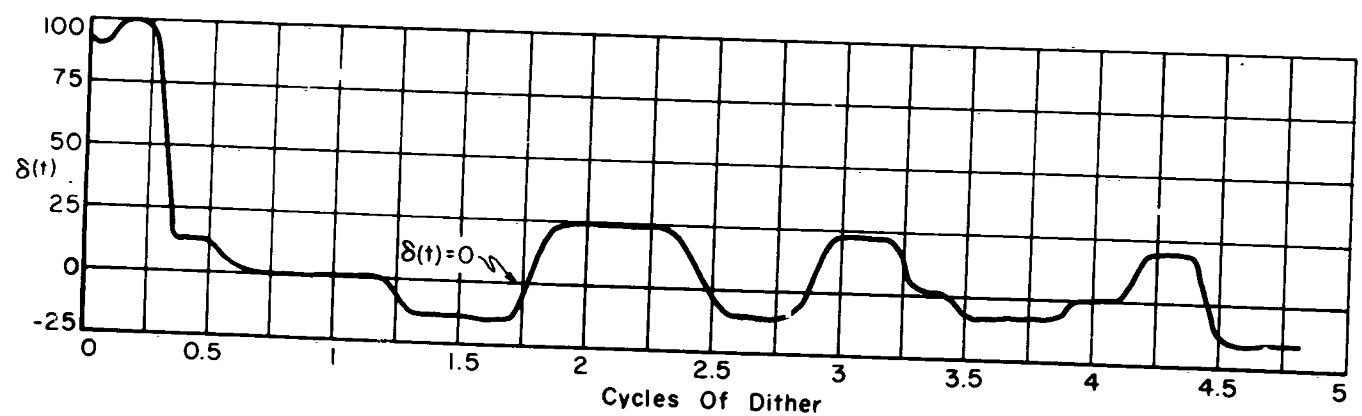

Fig. 8. (b) System of Figure 8 (b) with b_1 increased by factor of 4.

computer). It is significant that in five cycles of dither, the value of $\delta(t)$ is not significantly changed. This experimental observation supports the conclusion reached in Chapter III that when $\delta(t)$ is initially large, the error squared ($\epsilon^2{}_1$) is essentially independent of $\delta(t)$. If, by chance, through saturation of analog computer components, etc, the value of $\delta(t)$ decreases to some value that is small $(|\delta(t)| \ll |-a_2\omega^2 + j\,a_1\omega + a_o|)$, then from that point on, $\delta(t)$ will approach zero as before.

Figure 8(b) shows the behavior of $\delta(t)$ if, together with the large initial value of $\delta(t)$, b_1, the dither amplitude, is increased by a factor of four. The effect is that of increasing the dither to the point where the component of $\epsilon^2{}_1$ at the dither frequency is now significant with respect to the total error. Note that in this case the value of $\delta(t)$ drops to a small value during the first cycle of dither and thereafter oscillates about zero with a fairly large amplitude. This is a result of the fact that the component of $\epsilon^2{}_1$ at the dither frequency is now much larger than the error squared term, due to the input signal alone, and therefore predominates in the adaptive process. Although this situation has served to demonstrate a result of the analysis in Chapters III and IV, it is not satisfactory in general to allow the amplitude of the dither to be large with respect to the input signal.

One effective means of treating large initial values of $\delta(t)$ is

that of starting the adaptive process with large values of dither and then reducing the dither amplitude accordingly as $\delta(t)$ approaches zero.

To verify the conclusion made earlier that for large initial values of error, the system error squared is independent of the value of $\delta(t)$, the following experiment was performed: The circuit was opened at the output of the squaring device and this error squared was integrated over a constant time interval for different initial values of $\delta(t)$. The results are shown in graphical form in Figure 9. Note that as δ becomes larger than 2 (a relative value) the integral of error squared does in fact become independent of the amount of error, δ. This substantiates the conclusion reached at the end of Chapter IV.

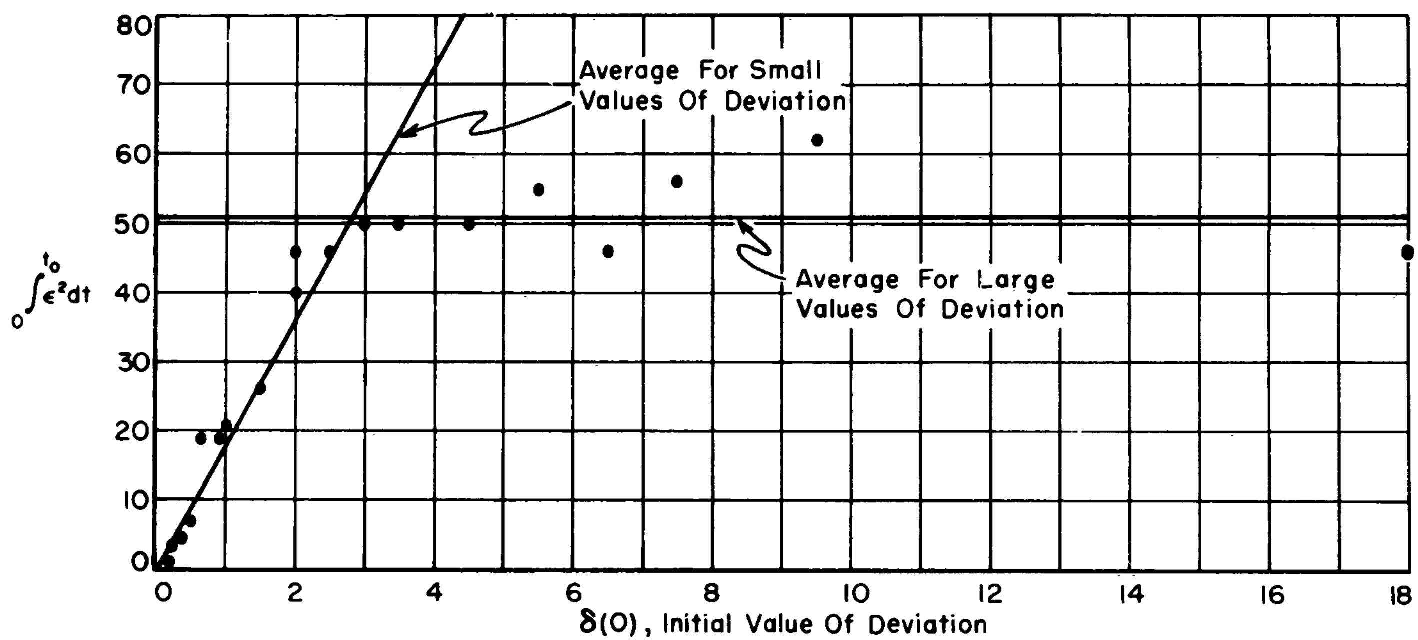

Fig. 9. Integral of error squared versus deviation.

CHAPTER VII

CONCLUSIONS

The material presented so far has consisted of the development of analysis methods and their application to a class of adaptive control systems. The anlysis of the slow system (Chapters III and IV) is relatively easy to perform, both because of the nature of the system and because of the resulting simplifications which can be made. In the analysis of the fast system, however, these assumptions are not applicable, and, for this reason, the analysis method used in Chapter V was devised. The analysis methods presented here are adequately suited to this class of adaptive control systems with respect to specifying the nature of the error signal (the performance criterion). Once this error signal is specified, conventional techniques may be used to evaluate system stability and performance.

Adaptive control systems have many varied applications. The slow system has been used to advantage in the determination of human parameters.[11] Here the model system was a human operator and the adjusted system was an analog computer representation of the human operator. The adaptive or supervisory loop adjusted the parameters in the analog computer to match those of the human, thus, in effect, measuring the human parameters. In an application of the fast system[14], radome error in an air-to-air system was

essentially cancelled by an adaptive control system in a small fraction of the flight time of the missile, improving missile performance greatly. This latter application does not belong strictly to the class studied in this paper, since its operation is dependent upon a sinusoidal test signal at the system input. Although both classes of adaptive systems have many applications, it is felt that the class considered in this paper is more versatile, in general. Other examples of applications of adaptive control may be listed as follows: The fast system could be used to optimize the bandwidth of the i-f section of a receiver based upon the maximization of the signal-to-noise ratio. Another example, which is in use at the present time, is an adaptive system that adjusts the gain of an automatic pilot system as a function of altitude and other environmental factors. Again, this latter application belongs to the class of systems having external test signals applied to the input.

Although adaptive control systems are built and used widely, there is still little known about synthesis procedures. The most common and perhaps the best synthesis tool today is the analog computer simulation. As systems become more complex, however, it is felt that the analog computer simulation will not be adequate for systems with large input signals or large parameter errors. It has been this author's experience that the slow system simulation, discussed in Chapter VII, will adapt even when the initial value of

the gain parameter $a_o + \delta(t)$ is negative. This, however, may be due to limiting or saturation or both in the analog computer, and mathematical techniques to handle cases of this type would certainly be desirable. When adaptive systems were first considered, it was thought by many that all the problems in the automatic controls field would be magically solved. However, this was certainly not the case; in fact, a host of new problems arose. Among these were the question of stability of the supervisory loop, the effect of dither signals upon the performance of the primary system, and others. Futhermore, there arose the question of specifying a performance criterion for adaptive systems. Although the supervisory loop operates on the basis of its own performance criterion, as soon as the system is in equilibrium, this criterion no longer exists. With this in mind, it is thought that the hunting loss as defined in the introductory paragraphs of Chapter V is an adequate criterion by which the performance of the adaptive system can be evaluated.

As more is learned about the analysis of these adaptive systems, it is felt that broad synthesis techniques will evolve. Until that time, however, the analog computer must be used, and new and better analysis techniques must be developed.

APPENDIX I

MATHEMATICS OF THE PHASE SENSITIVE DETECTOR

In the phase sensitive detector (Chapters III, IV, and V) the two input signals E_1 and E_2 are merely multiplied together to give E_o. Here E_1 is the square of a signal involving frequencies ω, $\omega + \omega_1$, and $\omega - \omega_1$. Let E_1 be given by

(A1-1)
$$E_1 = (a_1 e^{j\omega t} + a_{-1} e^{-j\omega t} + a_2 e^{j(\omega+\omega_1)t} + a_{-2} e^{-j(\omega+\omega_1)t} + a_3 e^{j(\omega-\omega_1)t} + a_{-3} e^{j(\omega-\omega_1)t})^2$$

where $\omega_1 << \omega$. Now writing E_1 at length, omitting all frequencies of $2\omega - 2\omega_1$ and higher,

(A1-2)
$$E_1 = a_1 a_{-1} + a_1 a_{-2} e^{-j\omega_1 t} + a_1 a_{-3} e^{j\omega_1 t} + a_{-1} a_2 e^{j\omega_1 t} + a_{-1} a_3 e^{-j\omega_1 t} + a_2 a_{-2} + a_2 a_{-3} e^{-2j\omega_1 t} + a_3 a_{-3} + a_{-2} a_3 e^{-j2\omega_1 t}$$

If this signal is now passed through the phase sensitive detector, the d. c. term of the output E_o will be given by the d. c. part of the product of E_1 and $(e^{j\omega_1 t} + e^{-j\omega_1 t})$

(A1-3)
$$E_o(dc) = a_1 a_{-2} + a_1 a_{-3} + a_{-1} a_2 + a_{-1} a_3 .$$

This is essentially the product of the $(\omega - \omega_1)$ frequency terms by the $(-\omega + \omega_1)$ terms, which is exactly the product formed in the earlier chapters. Thus, it is valid merely to drop the $e^{\pm j\omega_1 t}$ from the signal ϵ^2 in the earlier chapters to obtain the expression for the output of the phase sensitive detector since in the notation used there, both positive and negative frequencies, corresponding to $\pm\,\omega$, are included.

APPENDIX II

APPROXIMATIONS IN THE COMPLEX PLANE

In Chapter III, an approximation was made having the form

$$G - G_o = - \delta G_o^2$$

where

$$G = \frac{1}{-a_2 \omega^2 + j a_1 \omega + a_o + \delta}$$

and

$$G_o = \frac{1}{-a_2 \omega^2 + j a_2 \omega + a_o}$$

under the restriction that $\delta << \left| a_o + j a_1 \omega - a_2 \omega^2 \right|$.

It is clear that the approximation (for $\delta << 1$) of $\frac{1}{1+\delta} \cong 1 - \delta$ is valid when the variables are real; it remains to be shown that the approximation is valid when it later appears within a contour integral as shown below.

$$\text{(A2-1)} \qquad \int_C F(s)[G - G_o]\, ds = \int_C F(s)(-\delta G_o^2)\, ds.$$

Proving that the above are nearly equal is equivalent to proving the following

$$\text{(A2-2)} \qquad \int_C \left(\frac{f(s)}{s - s_o - \Delta s_o} - \frac{f(s)}{s - s_o} \right) ds \cong \int_C \frac{f(s)\Delta s_o}{(s - s_o)^2}$$

If one divides through both sides of the above by $2\pi i \Delta s_o$, then it is true that

$$\text{(A2-3)} \qquad \frac{1}{2\pi i \Delta s_o} \int_C f(s)\left[\left(\frac{1}{s - s_o - \Delta s_o}\right) - \frac{1}{s - s_o}\right] ds \overset{?}{=}$$

$$\frac{1}{2\pi i} \int_C \frac{f(s)}{(s - s_o)^2} ds$$

Now if $f(s)$ is analytic within the contour C, the left-hand side of the above equation is equal to (from the Cauchy Integral Theorem [4])

$$\text{(A2-4)} \qquad \frac{f(s_o + \Delta s_o) - f(s_o)}{\Delta s_o} = \frac{1}{2\pi i \Delta s_o} \int_C \left(\frac{1}{s - s_o - \Delta s_o} - \frac{1}{s - s_o}\right) f(s) ds$$

and the limit as $\Delta s_o \to 0$ of the above is

$$\text{(A2-5)} \qquad \lim_{\Delta s_o \to 0} \left(\frac{f(s_o + \Delta s_o) - f(s_o)}{\Delta s_o}\right) = f'(s_o) \, .$$

But the right-hand side of the above is given by

$$\text{(A2-6)} \qquad \frac{1}{2\pi i \Delta s_o} \int_C \left(\frac{1}{s - s_o - \Delta s_o} - \frac{1}{s - s_o}\right) f(s) ds \doteq \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - s_o)}$$

which also is equal to $f'(s_o)$.

It remains to be shown that the approximation is valid when Δs_o

is small but non-zero. Consider the difference between the two integrals

$$\text{(A2-7)} \quad \frac{1}{2\pi i \Delta s_o} \int_C \frac{\Delta s_o f(s)}{(s - s_o - \Delta s_o)(s - s_o)} ds - \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{(s - s_o)^2} ds$$

$$= \frac{1}{2\pi i} \int_C \frac{[(s - s_o)-(s - s_o - \Delta s_o)]\ f(s)\, ds}{(s - s_o)^2 (s - s_o - \Delta s_o)}$$

The right-hand side is written as

$$\text{(A2-8)} \qquad \frac{\Delta s_o}{2\pi i} \int_C \frac{f(s)\, ds}{(s - s_o)^2\ (s - s_o - \Delta s_o)}$$

If M is the maximum value of $f(s)$ on C and L is the length of C, and if d_o is the shortest distance from s_o to C and if $|\Delta s_o| < d_o$, then

$$\text{(A2-9)} \qquad \left| \Delta s_o \int \frac{f(s)\, ds}{(s - s_o)^2 (s - s_o - \Delta s_o)} \right| < \frac{ML\, |\Delta s_o|}{d_o^2 (d_o - |\Delta s_o|)}$$

The expression on the right is linear in Δs_o and therefore goes to zero as $\Delta s_o \to 0$. Now it is important to consider the case in which the contour C encloses a pole of $f(s)$. It is necessary to show the effect of the deviation Δs_o upon an integral around a different contour C' which encloses a pole of $f(s)$ but not s_o. Now it must be shown that

$$\text{(A2-10)}\quad \frac{1}{2\pi i\,\Delta s_o}\int_C\left(\frac{1}{s-s_o-\Delta s_o}-\frac{1}{s-s_o}\right)f(s)\,ds \cong \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-s_o)^2}$$

If $f(s)$ is expanded in partial fraction form, then let the pole which is contained in C' be $s = s_1$. Then the above integral may be written as

$$\text{(A2-11)}\quad \frac{1}{2\pi i}\int_C \frac{1}{(s-s_o)^2}\quad\frac{g(s)}{(s-s_1)}\; ds \overset{?}{=}$$

$$\frac{1}{2\pi i\Delta s_o}\int_C\left(\frac{1}{s-s_o-\Delta s_o}-\frac{1}{s-s_o}\right)\frac{g(s)}{s-s_1}\;ds\,.$$

The above approximate equation is evaluated, using the residue theorem, as

$$\text{(A2-12)}\quad \frac{g(s_1)}{(s_1-s_o)^2} \overset{?}{=} \frac{1}{\Delta s_o}\left(\frac{1}{s_1-s_o-\Delta s_o}-\frac{1}{s_1-s_o}\right) g(s_1)$$

or

$$\text{(A2-13)}\quad \frac{1}{(s_1-s_o)^2} \overset{?}{=} \frac{1}{\Delta s_o}\left(\frac{\Delta s_o}{(s_1-s_o-\Delta s_o)(s_1-s_o)}\right)\,.$$

Finally

$$\text{(A2-14)}\quad \frac{1}{(s_1-s_o)^2} \overset{?}{=} \frac{1}{s_1{}^2-2s_1s_o+s_o{}^2-\Delta s_o(s_1-s_o)}\,,$$

and since $|\Delta s_o|$ is small with respect to $|s_1 - s_o|$,

$$\text{(A2-15)} \qquad \frac{1}{(s_1 - s_o)^2} \cong \frac{1}{(s_1 - s_o)^2 - \Delta s_o(s_1 - s_o)}$$

Thus the two expressions are nearly equal.

It has been shown that the approximation made in Chapter III is a valid one when included in a contour integral, first, if the contour includes only the pole at which the approximation is made (s_o) and, second, if the contour includes only a pole of $f(s)$. Since the approximation is valid for both these cases, it is valid for any arbitrary contour.

APPENDIX III

NOTATION

The following symbols and notation are used throughout this paper

$a_o, a_1, a_2, \ldots$	- Constants .
b_1	- Dither signal amplitude .
$c_1, c_2, c_3, \ldots$	- Constants .
	- Complex representation of the error ϵ .
$F(s)$	- A function of the complex variable s .
$f(s)$	- A function of the complex variable s .
$G_o(j\omega) = G_o$	- Gain of the model system for input of frequency ω.
$G(jn\Delta) = G$	- Gain of the adjusted system for random input (t) .
$G(j(\omega + \omega_1)) = G_+$	- Gain of the system for input of frequency $\omega + \omega_1$.
G_{mn+}	- Gain of the system for input frequency $j[(m + n)\Delta + \omega_1]$.
I_n	- Solution of the integral of the form in Reference 3 .
$k_1, k_2, k_3, \ldots$	- Constants .
M	- A constant .
m, n	- Numbers which are almost but not quite equal to the integers [m], [n] . Normally the brackets are omitted when these numbers are used as integral indices. The ratio of any two, (e.g., m/n), is an irrational number.
x	- A random input of the form $x = \sum_n a_n e^{jn\Delta t}$.
y_m	- Output of the model system .
y	- Output of the adjusted system .
X	- Input expressed in complex form .
Y	- Output expressed in complex form .
$\sum_n$	- The sum over the infinite range of the index $-\infty < n < \infty$.
δ	- Small deviation of a parameter about its average value .
ϵ	- System error ; difference between outputs of model and adjusted system .

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I, Robert Bush Lackey, was born in Youngstown, Ohio, February 16, 1932. I received my secondary-school education in the Boardman public schools of Youngstown, Ohio. I received the combined degrees of Bachelor of Electrical Engineering and Master of Science from the Ohio State University in 1954. After two years as a Lieutenant (junior grade) in the United States Navy, where I served as Executive Officer, U.S.S. Whitehall, I was appointed Research Assistant in the Antenna Laboratory, Department of Electrical Engineering, The Ohio State University. In 1957, I was appointed Research Associate in the Antenna Laboratory and in 1959, Instructor in the Department of Electrical Engineering. I held these positions for two more years while completing the requirements for the Doctor of Philosophy degree.

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